Relational knowledge transfer with multiple agents

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Abstract

An expert specifies time paths of knowledge transfer and payments for two cash-constrained agents, who are free to walk away at any time, with the constraint that the expert can only train one agent during each period. The results show that in a profit-maximizing contrast, an agent is paid all previously accumulated wages in exchange for knowledge transfer during a period when he gets trained. Agents eventually receive all knowledge and have identical training duration. When players are not patient enough, the expert trains the two agents sequentially so that an agent is not trained until the training of the other agent is completed. When players are patient enough, the expert trains the two agents alternatively over time with similar time paths of knowledge transfer stocks. Training lasts longer when players are more patient, but the presence of other agents does not alter the training duration of each single agent.

JEL classification: C6; D8; J2; M5

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1. Introduction

1.1 Overview

During apprenticeships, agents often go through a stage in which they gradually acquire knowledge from their trainers while working long hours. As first noted by Becker (1964), the first-best allocation, which would involve transferring knowledge as quickly as possible, is not achievable since knowledge cannot be used as collateral. Moreover, when an expert has multiple agents to train during multiple periods, the expert faces the problem of allocating the amount of knowledge transfer and rewards among agents (spatial allocation), as well as designing the time schedule to train each agent (temporal allocation). This paper argues that under dynamic self-enforcing contracts with multiple agents, in which agents are trained gradually over time, (nearly) equal temporal and spatial allocation among agents can be both profit-maximizing and welfare-maximizing.

The model in this paper is an extension of that in Garicano and Rayo (2017), in which an expert (she) and two agents (he), both being risk-neutral, interact repeatedly over time. The expert has a stock of general purpose, perfectly divisible knowledge. The type of knowledge each agent acquires may be the same or different, and agents do not interact with each other. Initially, each agent has no knowledge and is not able to produce output. He also has no cash and thus cannot directly purchase knowledge from the expert. By transferring knowledge, the expert raises the agents' productivity and can extract from their output, but each agent may choose to leave with the knowledge already acquired and produce on his own.

Players rely on a self-enforcing multiperiod agreement, in which each agent may accept wages below output, but only to the extent that he is compensated with additional knowledge. The complication, compared to the single-agent case, is that the expert can only train one agent during a period (e.g., teaching some advanced skills requires one-to-one training). Effectively, the expert's problem is to design two contracts, but for the contract with each agent, there is a constraint that the expert cannot train the focal agent in certain time periods when the expert trains the other agents, and this constraint is chosen by the expert herself. However, an agent that is not trained during a period can still produce output and get paid. The expert needs to specify the time path for which agent to train, wages, and the amount of knowledge transfer for each agent.

I show that in a profit-maximizing contract, after an initial knowledge gift to initiate the generation of some output, each agent works for the expert, and may be paid or extracted money during periods when he is not trained, but the cumulative payments add up to 0 upon each period when he gets trained. In periods when the agent is trained, the value of the additional knowledge he receives is just high enough to compensate him for the current output he gives up and the cumulative wages he earns since the last periods when he is trained. The duration of the apprenticeship for an agent is determined by both the size of the initial knowledge gift, and the distribution of periods when the agent cannot be trained. These results are a generalization of those from the single-agent model in Garicano and Rayo (2017).

When designing the training schedule, the expert faces two trade-offs. First, by raising agents' productivity, a larger amount of the initial knowledge gift allows the expert to extract revenues more quickly from the agents, but also reduces the remaining knowledge that the expert can sell during the labor-for-training exchange. The other trade-off is that the expert wants to use future knowledge transfer to prevent both agents from leaving to have a high overall productivity, but he can only teach one agent during each period. More specifically, although training a focal agent over several consecutive periods can more quickly increase the productivity of the focal agent, it delays the training of the other agent, which lowers his productivity and can make it more difficult to prevent him from leaving.

I find that in a profit-maximizing contract, training is (nearly) alternative, that is, the expert takes turns transferring the two agents with the same amount of additional knowledge over time. Therefore, the knowledge stocks of the two agents grow in a parallel manner over time with a lag of only one period. Interestingly, the fact that there is more than one agent to be trained does not affect the duration of apprenticeship of each agent (i.e., the duration is the same as that in the single-agent case). Moreover, agents have the same length of duration, so that the agent who gets trained earlier will graduate earlier. As players become more patient, the apprenticeship gets longer and knowledge is transferred more slowly, since remaining knowledge becomes more valuable. I also show that every Pareto-efficient contract preserves the structure in the profit-maximizing contract, with the novice's Pareto-weight only reducing the duration of apprenticeship.

1.2 Related Literature

The current work is related to the literature on dynamic relational contracts between a principal and multiple agents, in which self-enforcing rewards motivate the agents (Calzolari and Spagnolo 2009; Board 2011; Andrews and Barron 2016; Deb et al. 2016; Ishihara 2017; Barron and Powell 2019; Kvaløy and Olsen 2019). This literature usually focuses on a different question about inducing effort exertion while treating the agents' productivity as fixed and exogenous. In contrast, the current paper assumes that agents always exert effort without cost, and investigates how the agents' productivity changes endogenously. Additionally, it is usually assumed that all multiple agents have the opportunity to participate in production and compete during each period, but in the current model, temporal asymmetry may occur.

For the human capital acquisition literature, many previous studies explain the incentives for firms to train agents by invoking market imperfection, such as uncertainty and asymmetric information about agents' quality (Katz and Ziderman 1990; Acemoglu and Pischke 1998), or matching frictions (Burdett and Smith 1996; Lowenstein and Spletzer 1998). This paper focuses on the dynamic self-enforcing mechanism, which was first proposed in Garicano and Rayo (2017). A following extension by Fudenberg and Rayo (2019) assumed that the agent can split effort between a knowledge-dependent "skilled task" and a knowledge-independent "unskilled task". As the apprenticeship proceeds, the extent of the agent's overwork decreases, and the agent spends a decreasing amount of time on menial effort. Recently, Fudenberg et al. (2021) introduced agent's effort exertion, and showed that a Pareto-efficient contract has an initial phase where the agent learns as fast as possible, followed by a longer phase during which the expert constrains the speed of knowledge transfer. However, all these models focus on the case when there is only a single agent and there is continuous training in every period. In contrast, when there are multiple agents, an agent cannot be trained continuously.

The rest of this paper is organized as follows. Section 2 presents the general model setup. Section 3 derives the properties of profit-maximizing contracts with a single agent under the constraint that the expert cannot transfer knowledge during some periods. Based on the results in Section 3, Sections 4 and 5 derives the profit-maximizing and Pareto-efficient contracts when there are two agents, respectively. Finally, I conclude and discuss the findings in Section 6.

2. The Baseline Model

I consider a baseline model with an expert (she) and N agents (he), all being risk-neutral. Players interact over infinite, discrete periods $t = 0,1, ...$ and discount future payoff using a common interest rate $r > 0$, with $\delta = \frac{1}{1+r}$ being the players' discount factor. The expert possesses one unit of general-purpose knowledge. The knowledge is perfectly divisible, does not depreciate, and can be transferred from the expert to the agent at any speed desired by the expert.

I use $X_{i,t} \in [0, 1]$ to denote agent i's stock of knowledge at the beginning of period t. Initially at $t = 0$, all agents have no knowledge (i.e., $X_{i,0} = 0, \forall i = 1,2, ..., N$). During each period t, the expert can only transfer knowledge to a single agent. The unit of a period (e.g., 1 hour, 1 day) can be interpreted as the duration of individual training, which may vary with the knowledge-transfer activities (e.g., a piano class takes about 1 hour, while teaching a molecular experiment may take one to several days). It should be emphasized that the discount factor δ should be larger when the time unit of a period is shorter.

During period t, each agent i works by himself and produces output $y_{i,t}= f(X_{i,t})$. I assume that the production function $f(\cdot)$ is continuous and increasing, with $f(0) = 0$. Therefore, in period 0, knowledge can be transferred but no output is produced. One interpretation of the production function is that an agent's output or performance is more valuable when he commands more knowledge or skills. Each period, the agent may choose to either remain in the contract and work for the expert, or leave the relationship and work for himself. Since knowledge is general, output is the same in both cases. I assume that an agent cannot return to reinvolve in the contract once he leaves. During each period t, the expert extracts profits $f\left(X_{i,t}\right)$ from each agent i's output, and compensates him by a monetary transfer $w_{i,t} \in \mathbb{R}$

(which I call a wage), and a transfer of additional knowledge $X_{i,t+1} - X_{i,t}$. I assume that agents cannot teach each other, so that the knowledge can only be transferred from the expert.

At the beginning of period 0, all players agree on a relational contract: a self-enforcing agreement that specifies a knowledge stock $X_{i,t}$ and wage $w_{i,t}$, for each period t and each agent i, conditional on the players remaining in the contract. I denote a relational contract by $\mathcal{C} = \left(\{X_{i,t}\}_{i=1}^{N}, \{w_{i,t}\}_{i=1}^{N}\right)_{t=0}^{\infty}$, which I call a contract for conciseness thereafter.

Let $\Pi_{i,t}(\mathcal{C})$ denote the expert's profits obtained from agent i, and $V_{i,t}(\mathcal{C})$ be agent i's continuation payoff from the standpoint of the beginning of period t. The expert's overall profits $\Pi_t(\mathcal{C})$ and agent i's payoff $V_{i,t}(\mathcal{C})$ in period t are, respectively,

$$\Pi_t(\mathcal{C}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} \Pi_{i,t}(\mathcal{C}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{i=1}^{N} \left[f\left(X_{i,\tau}\right) - w_{i,\tau}\right],$$

$$V_{i,t}(\mathcal{C}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} w_{i,\tau}.$$

At the beginning of period t, each player is free to walk away from the relationship, and the incentive compatibility constraints (IC) for the expert and agent i are, respectively,

$$\Pi_{i,t}(\mathcal{C}) \geq 0, \forall i, \forall t, \tag{1}$$

$$V_{i,t}(\mathcal{C}) \geq \frac{1}{1-\delta} f\left(X_{i,t}\right), \forall i, \forall t. \tag{2}$$

I assume agents have no access to credit and begin the relationship without any cash, thus the agents cannot simply buy all knowledge from the expert. As a result, the contract must also satisfy the liquidity constraint (LC) given by

$$\sum_{\tau=0}^{t} (1+r)^{t-\tau} w_{i,\tau} \geq 0, \forall i, \forall t\,. \tag{3}$$

Throughout the majority of this paper, I assume the expert has full bargaining power in the market, and thus she designs the contract $\mathcal{C}$ to maximize her profit. The expert's problem is

$$\max_{\mathcal{C}=\left(\{X_{i,t}\}_{i=1}^{N},\{w_{i,t}\}_{i=1}^{N}\right)_{t=0}^{\infty}} \Pi_0(\mathcal{C}), \tag{I}$$

subject to (1), (2), (3),

plus the two constraints that $X_{i,t} \in [0,1]$ is non-decreasing, and only a single agent may be taught during each period (i.e., $\left|\left\{i | i \in \{1,2,\ldots,n\}, X_{i,t+1} - X_{i,t} > 0\right\}\right| \in \{0,1\}$, $\forall t$). I also study the broader set of Pareto-efficient contracts that maximize a weighted sum of all players' payoffs.

Finally, following Garicano and Rayo (2017), if the expert completes the knowledge transfer to an agent i in finite time, I say that agent i graduates. I use T_i to denote the first period after which there is no longer knowledge transfer to agent i, so $T_i = \min\{t | X_{i,t} = X_{i,\tau}, \forall \tau \geq t\}$.

Since the agents do not interact with each other in terms of production and knowledge transfer, and each agent only cares about his own utility, a contract with N agents can be effectively considered as N contracts between the expert and each individual agent. However, each of the N contracts differs from that in Garicano and Rayo (2017) due to the fact that the expert is not able to transfer knowledge to the focal agent in some periods. In other words, the friction is that training a focal agent during a period inhibits training other agents.

Following the above conjecture, I derive the profit-maximizing contract in two steps. First, in Section 3, I present the properties of a profit-maximizing contract with a single agent, given the constraint that the expert cannot transfer knowledge during some periods. The result allows me to calculate the expert's profits obtained from each agent in a contract with multiple agents, based on which I then derive the profit-maximizing contract in Section 4.

3. Profit-maximizing Contracts with A Single Agent given Non-transfer Periods

In this section, I consider a contract with a single agent, and denote the agent's knowledge stock and wage by X_t and w_t, respectively. The model setup is similar to the baseline model in Garicano and Rayo (2017), with the additional constraint of a set of periods $\mathcal{B}$ during which knowledge transfer is not allowed, which I refer to as non-transfer periods. For ease of description, I denote the set of periods during which knowledge transfer is allowed as $\mathcal{A}$ ($\mathcal{A} = \mathbb{N}_+ \backslash \mathcal{B}$), which I refer to as the knowledge-transfer periods. The constraint of non-transfer periods can be expressed as

$$X_{t+1} = X_t, \forall t \in \mathcal{B}. \tag{4}$$

Therefore, the expert's problem is

$$\max_{\mathcal{C}=(X_t, w_t)_{t=0}^{\infty}} \Pi_0(\mathcal{C}), \tag{II}$$

subject to (1), (2), (3), (4).

It turns out that training will be completed in a finite time. Before graduation, upon every knowledge-transfer period, the agent earns zero cumulative wages and is compensated through additional training.

Proposition 1: *For a contract with a single agent, given the constraint of non-transfer periods $\mathcal{B}$, every optimal contract follows the following properties*:

(i) The agent graduates in finite time, and the knowledge transfer is complete, that is, $X_T = 1$ for some T.

(ii) The agent's liquidity constraint binds at every knowledge-transfer period before graduation. Namely, $\sum_{\tau=0}^{t} \delta^{\tau-t} w_\tau = 0$, for all $t \in \mathcal{A} \cap \{t \in \mathbb{N}_+ | t < T\}$. This means that the cumulative wages between two knowledge-transfer periods add up to 0, and the agent earns zero cumulative wages upon graduation. In other words, the value of additional knowledge transferred is just high enough to compensate the agent for the current output he gives up and the wage he earned since the last period when he was trained.

The intuition is similar to that given in Garicano and Rayo (2017). For Property (i), note that from any date onward, the overall profits the expert can extract from the agent are no greater than the value of the knowledge remaining to be transferred, and this value will approach zero as time proceeds due to discounting. Therefore, once the agent's output during a period exceeds the value of remaining knowledge, the expert will end the contract by selling all remaining knowledge at once. Moreover, since any additional knowledge has a positive value, the expert profits from selling all knowledge to the agent.

For Property (ii), suppose in an optimal contract with graduation date T, the agent's liquidity constraint does not bind at 0 at some knowledge-transfer periods before his graduation. Consider an alternative contract in which the agent graduates earlier at $T' < T$ but earns zero cumulative wages before graduation (after T' the wage is $f(1)$), with T' chosen such that the agent's cumulative wages in the present value, thus his payoff, do not change. Since now the agent must wait longer to earn his wages, his continuation values V_t increase. Since now the incentive constraints do not bind, the expert can increase the agent's payoff unchanged.

Proposition 1 suggests that to find the optimal contract with a focal agent given certain non-transfer periods, it is sufficient to focus on contracts that satisfy the properties in Proposition 1. Denote the knowledge-transfer periods before graduation by $\{t_k\}_{k=1}^{K} = \mathcal{A} \cap \{t | t < T\}$, where K is the number of knowledge-transfer periods. The initial and the last knowledge-transfer periods are $t_1 = 0$ and $t_K = T - 1$, respectively. Proposition 2 characterizes the expert's knowledge transfer schedule over time in a contract that satisfies the properties in Proposition 1.

Proposition 2: *Given the constraint of non-transferable periods $\mathcal{B}$, let $\mathcal{C}$ be a feasible single-agent contract (satisfying constraint (1)-(4)) that has graduation date T and satisfies the properties in Proposition 1. Before graduation, the agent's knowledge stock after each knowledge-transfer period is*

$$f(X_{t+1}) = \delta^{T-(t+1)} f(1), \forall t \in \mathcal{A}, t < T.$$

Corresponding, the expert's profits at $t = 0$ are

$$\Pi_0(\mathcal{C}) = \frac{\sum_{k=1}^{K-1}(1 - \delta^{t_{k+1}-t_k})}{1 - \delta} \delta^T f(1). \quad (5)$$

Proposition 2 shows that after each knowledge-transfer period, the agent's knowledge stock depends only on the time length towards graduation, but not on the non-transfer periods. As a result, from the standpoint of period 0, the profits the expert obtains from agent i after every knowledge-transfer period are always equal to δ^{T_i}, where T_i is the graduation date. However, during non-transfer periods between two knowledge-transfer periods, the profits obtained during each non-transfer period will decrease over time with a rate of δ. Therefore, as equation (5) indicates, the expert's profits obtained from the agent only depend on the distribution of the time length between two knowledge-transfer periods (i.e., $\{t_{k+1} - t_k\}_{k=1}^{K-1}$), which leads to an important property described in Corollary 1.

Corollary 1: Consider contracts that satisfy the properties in Proposition 2 for each agent. Let $\mathcal{C}$ be a contract in which the expert trains agent i during periods $\underline{t}$ and $\bar{t}$ ($\underline{t} < \bar{t}$), and agent $j \neq i$ gets trained before $\underline{t}$ and graduates after $\bar{t}$, (i.e., $T_j > \bar{t}$). Let $\{t_{j,k}\}_{k=k^*}^{k^{**}} = \{t_{j,k}\}_{k=1}^{K_2} \cap \{t | \underline{t} < t < \bar{t}\}$ be the periods when the expert trains agent j during the range $\underline{t} < t < \bar{t}$. It does not change the expert's profits obtained from agents i by increasing or decreasing $\{t_{j,k}\}_{k=k^*}^{k^{**}}$ by the same amount of periods such that $\{t_{j,k}\}_{k=k^*}^{k^{**}}$ are still in the range $\underline{t} < t < \bar{t}$.

For intuition, note that the expert's profits only depend on the distribution of the time length between two knowledge-transfer periods. Clearly, the modification does not change the distribution for agent i, which is the periods during which the expert trains agent j.

4. Profit-maximizing Contracts with Two Agents

In this section, I return to the original problem proposed in section 2, which is to find the optimal contract with multiple agents. To derive the optimal contract, it is sufficient to focus on contracts in which knowledge transfer occurs in every period before all agents graduate. Otherwise, suppose there is a period when no agent gets trained; the expert can increase her profits by training any agent during that period, which increases his knowledge stocks and productivity. Moreover, for a focal agent, conditioned on the expert's contracts with other agents (thus the non-transfer periods for the focal agent are given), the optimal contract with the focal agent should satisfy the properties in Propositions 1 and 2, which determine the expert's profits obtained from the focal agent. This is also true for the contracts with other agents. Therefore, we only need to focus on the case in which the expert's contract with every agent satisfies the properties in Propositions 1 and 2.

For simplicity and better intuition, I focus on the case of two agents. I denote the agent who gets trained first in period 0 by 1 and the other by 2. I denote the graduation date of agent $i \in \{1,2\}$ by T_i, and let T_{max} be the maximum graduation date $\max\{(T_i)_{i=1}^N\}$. I denote the sequence of the periods during which agent i is trained before he graduates by $\{t_{i,k}\}_{k=1}^{K_i}$.

The contracts with two agents can be classified into two scenarios, depending on whether agent 2 gets trained before agent 1 graduates or not[1]. I first characterize the optimal contracts under the scenario when agent 2 gets trained only after agent 1 graduates.

4.1. Agent 2 is not trained until agent 1 graduates

Lemma 1: *Up to an integer constraint of* $T^* = 1 - \frac{1}{\ln\delta}$ *and* $\frac{\delta}{e\ln\delta}$*, for every optimal contract* $\mathcal{C}$ *in which agent 2 is not trained before agent 1 graduates, the graduation dates of agents 1 and 2 are* $T_1 = T^* + \frac{\delta}{e\ln\delta}$*, and* $T_2 = 2T^* + \frac{\delta}{e\ln\delta}$*, respectively, with payments and knowledge transfer to each agent satisfying the properties in Propositions 1 and 2.*

Proof: Clearly, it is optimal for the expert to train agent 2 immediately after agent 1 graduates, and since now there is only a single agent, the optimal duration of training for agent 2 is T^* (thus $T_2 = T_1 + T^*$), as given by Garicano and Rayo (2017). Therefore, the expert's profits are a function of agent 1's graduation date T_1 as $\left[T_1 - 1 + \delta^{T^*}(T^* - 1)\right]f(1)$. The first-order condition with respect to T_1 shows that at the optimality[2], $T_1 = T^* + \frac{\delta}{e\ln\delta}$. ■

Since $T^* + \frac{\delta}{e\ln\delta} < T^*$, for a contract with multiple agents in which each agent is trained continuously until his graduation, the agent who is trained earlier will have shorter a graduation date than the optimal graduation date under the case when there is only a single agent, T^*. Intuitively, the expert gives up some profits obtained from agent 1 by speeding up knowledge transfer in order to start training agent 2 earlier.

4.2. Agent 2 gets trained before agent 1 graduates

For the case when agent 2 gets trained before agent 1 graduates, the derivation is less intuitive. In this scenario, the expert earns no profits when $T_{max} \leq 2$. When $T_{max} = 3$, the expert cannot obtain

[1]Note that agent 2 may or may not get trained in the scenario when agent 2 is not trained before agent 1 graduates.

[2]The first-order condition is $\delta^{T_1}\left[1 - \frac{\delta}{e} + (T_1 - 1)\ln\delta\right]f(1) = 0$.

profits from agent 2, so the expert's profits are higher when she continuously trains agent 1[3]. Therefore, it is sufficient to focus on the case when the maximum graduation date is greater than 3 (i.e., $T_{max} \geq 4$). Lemma 2 shows that in order to derive the optimal contract, it is sufficient to focus on contracts a special knowledge transfer schedule, as illustrated in Figure 1.

Lemma 2: *For any contract* $\mathcal{C} = \left(\{y_{1,t}, y_{2,t}\}, \{w_{1,t}, w_{2,t}\}\right)_{t=0}^{\infty}$ *with* $T_{max} \geq 4$, *in which the contract with each agent satisfies the properties in Propositions 1 and 2, there exists an alternative contract* $\mathcal{C}' \in \mathcal{H}$ *that gives the same or higher profits to the expert, where* $\mathcal{H}$ *is a set of contracts with the knowledge transfer schedule being as follows:*

(*i*) *Begins with* $n \geq 1$ *rounds of alternate training between the two agents. Namely, agent 1 is trained in periods* $t = 0, 2, \dots, 2n-2$, *and agent 2 is trained in periods* $t = 1, 3, \dots, 2n-1$.

(*ii*) *Followed by* $m = T_j - 2n - 1$ ($j \in \{1,2\}$) *consecutive knowledge-transfer periods for one of the agents. Namely, agent 1 or 2 is trained in periods* $t = 2n, 2n+1, \dots, T_j - 2$.

(*iii*) *Agent* j *is trained in period* $t = T_j - 1$, *and thus graduates at* $t = T_j$.

(*iv*) *Followed by* $l + 1 = T_i - T_j$, ($i \in \{1,2\}$ *and* $i \neq j$) *consecutive knowledge-transfer periods for agent* i. *Namely, agent* i *is trained in periods* $t = T_j, T_j + 1, \dots, T_i - 1$.

(*v*) *The contract with each agent satisfies the properties in Proposition 1 and 2.*

Proof: First I show that if an agent is trained in two consecutive periods t' and $t'+1$ (without loss of generality, say it is agent 1), the expert may increase her profits by moving all knowledge-transfer periods for agent 2 during period t' and T_1 earlier by a same amount such that $t_{2,1} = t'+1$. Corollary 1 suggests that this modification does not change the expert's profits obtained from agent 1. If agent 2 graduates before T_1, this modification increases the expert's profits obtained from agent 2 due to both quicker and earlier training. If agent 2 graduates after T_1, the modification increases the time interval between the last knowledge-transfer period of agent 2 before T_1 and the first knowledge-transfer period after T_1, thus

[3]When $T_{max} = 3$, given agent 2 gets trained before agent 1 graduate, agent 1 is trained at $t = 0$, 2, and agent 2 is trained during $t = 1$. The profits are lower than agent 1 is trained during $t = 0,1,2$.

increasing the profits obtained from agent 2 during the two knowledge-transfer periods. Also, the modification does not change the profits obtained in other periods. If agent 2 gets trained before t', by Corollary 1, this modification does not change the distribution of non-transfer interval for agent 2. If agent 2 gets trained after t', denote the first knowledge-transfer period of agent 2 by $t_{2,1}$, this modification increases the duration of training for agent 2, $T_2 - t_{2,1}$, by $t_{2,1} - (t' + 1)$, which discounts the knowledge stock of agent 2 in each period before T_1 by $\delta^{t_{2,1}-(t'+1)}$. However, since agent 2 is trained earlier by a time length of $t_{2,1} - (t' + 1)$, in terms of expert's profits, the two opposing effects cancel each other out. The same logic applies to the case when agent 2 is trained in two consecutive periods.

Now consider a contract $\mathcal{C}$ that satisfies the properties in Propositions 1 and 2, with $t_{1,1} = 0$ and graduation dates T_1 and T_2. Let $T_{min} = \min\{T_1, T_2\}$. A contract $\mathcal{C}' \in \mathcal{H}$ which gives equal or higher profits than $\mathcal{C}$ can be obtained based on the following the procedure:

(i) Let $i = 1$.

(ii) If $t_{2,i} > 2i - 1$, reduce all the knowledge-transfer periods of agent 2 between periods $t_{2,i}$ and $T_{min} - 1$ by $t_{2,i} - (2i - 1)$ so that $t_{2,i} = 2i - 1$. The expert trains agent 1 in the rest periods before period $T_{min} - 1$. Adjust wages and knowledge stocks so that the new contract satisfies the properties in Propositions 1 and 2.

(iii) If $t_{1,i+1} > 2(i + 1)$, reduce all the knowledge-transfer periods between periods $t_{1,i+1}$ and $T_{min} - 1$ by $t_{1,i+1} - 2(i + 1)$ so that $t_{1,i+1} = 2(i + 1)$, and train agent 2 in the rest periods before $T_{min} - 1$. Adjust wages and knowledge stocks so that the new contract satisfies the properties in Proposition 1 and 2.

(iv) Let $i = i + 1$, and repeat steps (ii)-(iv) until $t_{1,i+1} \geq T_{min} - 1$ or $t_{2,i} \geq T_{min} - 1$. ■

Lemma 2 allows us to focus on a subset of contracts and obtain an analytical expression of the expert's profits based on the special properties of these contracts. We can then characterize the properties of the optimal contracts under the scenario when agent 2 gets trained before agent 1 graduates.

Lemma 3: *Up to an integer constraint of* $T^* = 1 - \frac{1}{\ln \delta}$ *with* $T^* \geq 3$, *every optimal contract* $\mathcal{C}$ *in which agent 2 is trained before agent 1 graduates satisfies the following properties:*

(*i*) *Agent 2 graduates right after agent 1 graduates* (*i.e.*, $T_2 = T_1 + 1$) *and* $T_1 = T^*$.

(*ii*) *The expert alternates training between agents 1 and 2 as much as possible* (*i.e.*, $m \in \{0,1\}$). *When* T^* *is even so that* $m = 1$, *agent 1 is trained during period* $t = 2n$.

(*iii*) *Wages and knowledge transferred to each agent satisfy the properties in Propositions 1 and 2. That is, each agent graduates in finite time* T_i *with all knowledge transferred, earns zero cumulative wages upon each period he gets trained. Before graduation, his knowledge stock after a training period* t *is* $f(X_{t+1}) = \delta^{T_i-(t+1)} f(1)$.

The proof sketch and intuition are as follows:

(i) Agent 1 graduates before agent 2: Suppose agent 1 graduates after agent 2 ($T_1 > T_2$), the expert can increase her profits by swapping the graduation dates of the two agents (i.e., let $T_1' = T_2$, $T_2' = T_1$), while keeping the training schedules by period T_1' to be the same. Before the modification, agent 2's productivity grows fast while agent 1's productivity grows slowly, and the modification roughly swaps the two dynamics of productivity. Before the modification, agent 2 is trained later than agent 1, so the fast-growing productivity dynamics start later than the slow-growing productivity dynamics. The modification increases the expert's profits by starting the fast-growing productivity dynamics earlier.

(ii) Agent 1 is trained during the m *consecutive periods after alternate training*: Recall that Proposition 2 shows that the knowledge stock (thus productivity) during a period only depends on the distance between the focal period to the graduation date. Given that agent 1 graduates first, if agent 1 is trained during the m consecutive periods, his productivity during these periods will be higher than agent 2's productivity if these periods are used to train agent 2.

(iii) The expert alternates training between two agents as much as possible before agent 1 graduates: Proposition 2 shows that the profits obtained from an agent i are always δ^{T_i} after each knowledge-transfer period, but the profits will decrease at a rate δ over time during non-transfer periods

due to discounting. If the expert reduces the rounds of alternative training from n to $n-1$, agent 1 will replace agent 2 to receive training during period $2n-1$ in Figure 1. This modification increases the expert's profits obtained from agent 1 by $(1-\delta)\delta^{T_1}$ due to an increase in the productivity of agent 1 during period $2n$. However, since at optimality, agent 1 is trained during the m consecutive periods after alternative training, this modification increases the duration of non-transfer periods of agent 2 from $m+1$ to $m+3$, and thus systematically reduces the productivity of agent 2 in every period from period $2n$ to period T_1. This causes a total loss of the profits obtained from agent 2 $\delta^{T_2}(1+\delta)(1-\delta^m)$, which outweighs the increased profits from agent 1.

(iv) Agent 2 graduates right after agent 1 graduates: The previous steps show that in an optimal contract, agent 1 should graduate first, and before his graduation, the expert alternates training as much as possible. Therefore, by Proposition 2, the knowledge stocks of both agents 1 and 2 grow in a nearly constant rate over time before agent 1's graduation date (increased by $1/\delta^2$ every 2 periods). Since agent 2's knowledge stock dynamics is nearly identical to agent 1's dynamics, with a delay of 1 period, the graduation date of the two agents should be the same, which means gent 2 should graduate right after agent 1 graduates.

Lemma 3 shows that the optimal training duration of each agent, $1-\frac{1}{\ln\delta}$, is the same as when there is only one agent (Garicano and Rayo 2017). In other words, even when the expert can only teach one agent during a period, the addition of other agents does not affect each agent's training duration. This is because when the expert trains the two agents alternatively, the sum of the productivity of agents 1 and 2 grows at a constant rate over time as it does in the single-agent case, as illustrated by Figure 2. Therefore, the contract with two agents can be effectively considered as a single-agent contract.

It should be noted that whether fully alternative training is implementable depends on whether T^* is odd or even. Training will be fully alternative when T^* is odd. When T^* is even, fully alternate training is not possible, and agent 1 will be trained consecutively in periods T^*-2 and T^*-1.

4.3. Optimal contracts

Lemmas 1 and 3 enable us to calculate the expert's maximum profits under the two scenarios when agent 2 is not trained before agent 1 graduates and is trained before agent 1 graduates, respectively. The optimal contract is then obtained by comparing the maximum profits between these two scenarios. Proposition 3 relaxes the integer constraint assumed in Lemmas 1 and 3, and shows that alternate training is optimal when players are patient enough.

Proposition 3: *In an optimal contract with two agents that solves problem (I), when* $0 < \delta < \delta^*$*, where* $\delta^* \approx 0.555$ *solves the equation* $\delta^3 - 2\delta^2 - \delta + 1 = 0$*, the expert trains agents sequentially (i.e., she starts training agent 2 only after agent 1 graduates). Agent 1's graduation date is 2, and agent 2's graduation date is 4 when* $\delta < 0.5$ *and 5 when* $0.5 < \delta < \delta^*$*. For* $\delta^* < \delta < 1$*, the expert alternates training between agents as much as possible, and agent 2 graduates right after agent 1's graduation. Let* $\underline{T} = \text{floor}\left(1 - \frac{1}{\ln \delta}\right)$*. When* $\underline{T}$ *is even, agent 1's graduation date is* $\underline{T}$ *when* $\underline{T} > \frac{2\delta(2-\delta)}{1-\delta^2}$*, and* $\underline{T} + 1$ *when* $\underline{T} < \frac{2\delta(2-\delta)}{1-\delta^2}$*. When* $\underline{T}$ *is odd, agent 1's graduation date is* $\underline{T} + 1$ *when* $\underline{T} > \frac{1+\delta(2+2\delta^2-3\delta)}{1-\delta^2}$*, and* $\underline{T}$ *when* $\underline{T} < \frac{1+\delta(2+2\delta^2-3\delta)}{1-\delta^2}$.

Proposition 3 shows that the optimal training has two contrasting types of dynamics depending on the discounting factor. Note that initially, agents have no knowledge, so the expert must transfer some "knowledge gift" for free to initiate production. Therefore, to earn profits, training must last for at least two periods. Since agents still produce output during periods when they are not trained, overall productivity will be higher if more agents remain in the contract. Therefore, the expert wants to use future additional knowledge transfer to prevent agents from leaving. However, since the expert can only train one agent during each period, this means that some of the agents must wait for at least one period to get trained, which makes it harder to prevent them from leaving.

When players are not patient enough, the optimal training is sequential. Suppose the expert does not train agent 1 after transferring the initial knowledge gift. Since agent 1 is not patient, he values his current output more and cares little about additional knowledge transfer that increases his productivity in the future. Therefore, he cannot wait for even as short as one period to receive training, and would rather leave with his current knowledge and work by himself after the initial knowledge gift. As a result, it is better for the expert to complete training an agent once it starts training him, and then starts training other agents. Since initially, all agents have no knowledge, the expert can start training an agent at any time. When players are patient enough, both agents value the additional knowledge transfer in the future, and can wait for 1 or even more periods to receive training, allowing alternate training to be implementable.

Although Lemmas 1 and 3 assume an integer constraint for the optimal graduation date $1-\frac{1}{\ln \delta}$, in the proof of Lemmas 1 and 3, the discounting factor is treated as a continuous variable. Therefore, as the last part of Proposition 3 shows, relaxation of the integer constraint does not change the overall structure of the optimal contract, but may only affect the graduation date by 1 period since now the optimal graduation date may not be an integer.

A small discount factor can occur if each training period takes a long time (e.g., training a big project), or if the relative productivity of the knowledge declines quickly over time. The latter may occur when the knowledge has short-term effectiveness with respect to productivity, which may be because the knowledge is likely to quickly become outdated, or lose its monopoly status in the future. In these scenarios, Proposition 3 predicts that training should be sequential and completed in a short time, so that only one agent will be trained during a certain time range. On the other hand, Proposition 3 predicts that multiple agents will be trained within a certain time range if each training period is short (e.g., a piano lesson, or a homework problem), or if the expert's knowledge has long-term effectiveness in productivity.

5. Pareto-efficient Contracts

Here I characterize the broader set of Pareto-efficient contracts, by solving the problem of a Planner who maximizes a weighted sum of the players' payoff,

$$\lambda \sum_{i=1}^{N} V_{i,0}(\mathcal{C}) + \Pi_0(\mathcal{C}) = \sum_{i=1}^{n} \left(\lambda V_{i,0} + \Pi_{i,0}\right), \tag{III}$$

subject to the same constraints as those in problem (I). The parameter $\lambda \geqslant 0$ is the agents' Pareto weight, which is assumed to be identical for all agents. Similar to that in section 4, I focus on the case when there are two agents.

Corollary 2: *Suppose $\mathcal{C}$ is a Pareto-efficient contract $\mathcal{C}$ with two agents that solves the Planner's problem (III) for a given weight λ. Then,*

(*i*) *$\mathcal{C}$ has all properties in Proposition 1 and 2. Namely, each agent graduates at a finite date T with complete knowledge, and earns zero cumulative wages upon each period when he is trained. The agent's knowledge stock after each period t when he is trained is $f(X_{t+1}) = \delta^{T-(t+1)} f(1)$.*

(*ii*) *Let $T_\lambda^* = 1 - \frac{1}{\ln \delta} - \frac{2\lambda}{1-\delta^2} \geq 3$. Up to an integer constraint of T_λ^* with $T_\lambda^* \geq 3$, the expert trains the two agents alternately as much as possible, and the duration of training for each agent is T_λ^*.*

The intuition of Corollary 2 is similar to that in the proof of the properties for a profit-maximizing contract. Corollary 2 means that every Pareto-efficient contract preserves the structure of profit-maximizing contracts except for the agents' date of graduation T. Unlike that in a profit-maximizing contract, where the training duration of each agent is the same as that when there is only a single agent. For Pareto-efficient contract, the efficient graduation duration date is influenced by the presence of other agents, as the efficient training duration when there is only a single agent is $1 - \frac{1}{\ln \delta} - \frac{\lambda}{1-\delta}$, but the difference between the two values is very slight.

6. Conclusion

Briefly, I have considered the optimal contract for multiperiod training arrangement between an expert and multiple agents, where the expert with commitment power to sell her knowledge to cash-constraint agents. The expert faces the constraint that she can train only one agent in each period, which may occur when professional skills or knowledge require meticulous instructions and must be taught one-on-one (e.g., more advanced skills). In the optimal contract, each agent receives no cumulative wages upon the periods when he gets trained before graduation. When players are impatient, agents are trained sequentially. That is, the expert trains an agent only after she finishes training the previous agent, but agents graduate quickly in 2 or 3 periods. When players are patient enough, the duration of training becomes longer, and in the optimal contract, the expert trains agents (nearly) alternately with the same amount of additional knowledge transfer. Compared to a contract with a single agent, the presence of other agents does not affect the training duration of each agent.

Alternate training is prevalent in daily life, and the present results offer a possible explanation based on individual rationality. Another explanation for alternate training is inequity aversion. If agents dislike unequal allocations of time and the amount of knowledge transfer, expecting this, the expert may be motivated to allocate alternate training.

Finally, the present model mainly focuses on the case of two agents, and an extension of the model for any finite number of agents is needed, but it is natural to expect that the results may preserve the overall structure of the current findings. The present model assumes that agents do not interact, so it may be interesting to explore the case when agents with more knowledge can train others with less knowledge. Additionally, the current model assumes that all players are involved in the relationship at the same time, and it may be valuable to investigate the case when some agents join the contract later. In this scenario, the expert may want to quickly bridge the knowledge gap between agents. Furthermore, when agents can join later, the expert may also maintain a constant pool of agents in a balancing state between recruiting new agents and completing the training of current agents.

Figure

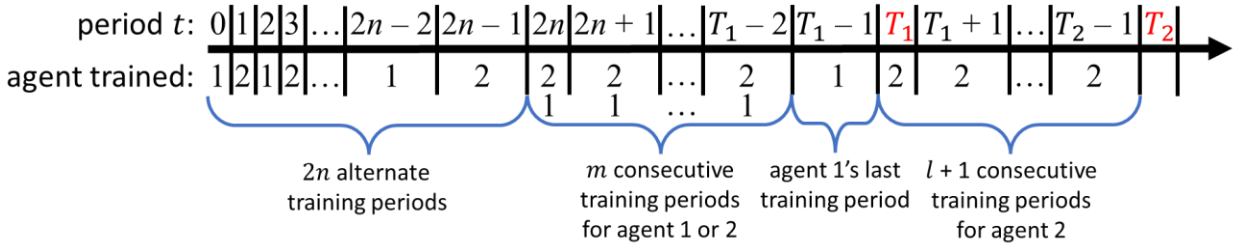

Figure 1. An illustration of the knowledge transfer schedule in a contract in the set $\mathcal{H}$ with properties described in Lemma 2 when agent 2 graduates later than agent 1 ($T_1 < T_2$).

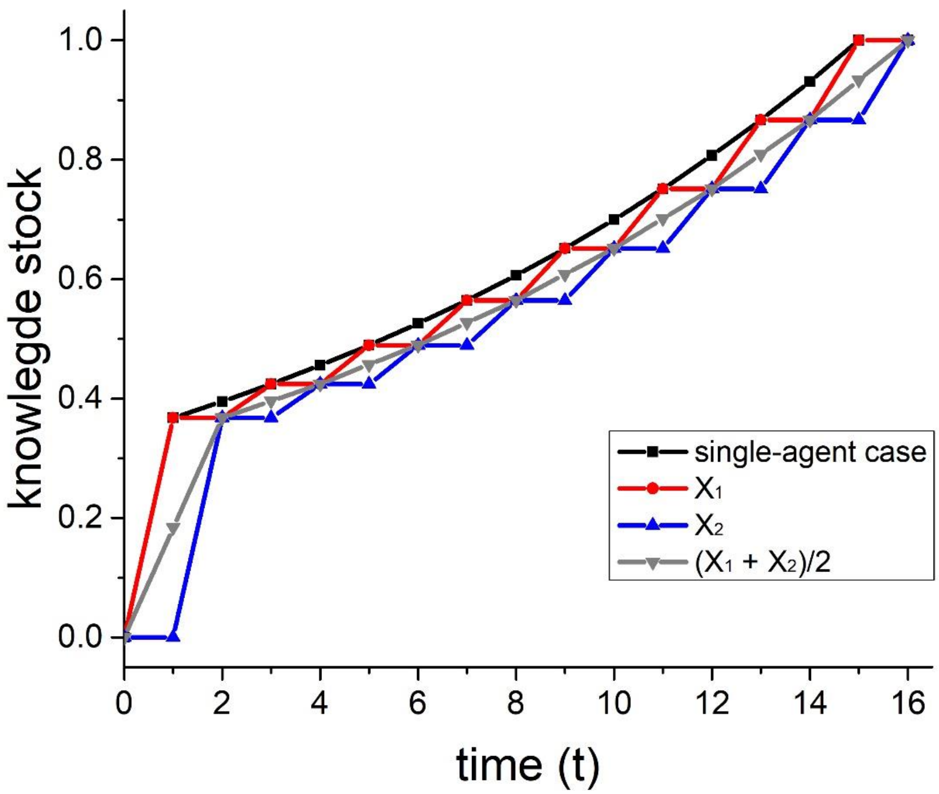

Figure 2. Growth of knowledge stock over time of agent 1 (X_1) and agent 2 (X_2) and the average $\frac{X_1+X_2}{2}$ under alternate training when $T_1 = 15$, $T_2 = 16$. The black line shows the optimal knowledge stock dynamics in a single-agent contract with graduation date being 15.

7. Appendix

Proof of Proposition 1:

Property (*i*): When $\mathcal{A}$ is finite, T must be finite. Therefore, it is sufficient to focus on the case when the set of knowledge-transfer periods $\mathcal{A}$ is infinite. Suppose that a contract $\mathcal{C} = (y_t, w_t)_{t=0}^{\infty}$ is optimal (i.e., it solves problem (II)) but the training takes infinitely long, that is, $y_t < \overline{y}$ for all t, where $\overline{y} = \lim_{t\to\infty} y_t$. Since $\mathcal{A}$ is infinite, there exists a large enough knowledge-transfer period $k \in \mathcal{A}$ such that $y_k \geq \frac{1}{r}(\overline{y} - y_k)$ and consider a new contract $\mathcal{C}' = (y_t', w_t')_{t=0}^{\infty}$ with $y_t' = w_t' = \overline{y}$ for all $t > k$, and $w_k' = V_k(\mathcal{C}) - \frac{1}{r}\overline{y} \geq 0$. In other words, in period k, the agent gives up net output $y_k - w_k'$ in exchange for additional knowledge transfer $\overline{y} - y_k$. Therefore, $\mathcal{C}'$ delivers a strictly higher profit than $\mathcal{C}$, a contradiction.

Property (*ii*): Denote the sequence of knowledge-transfer periods before the graduation date T by $\{t_k\}_{k=1}^{K} = \mathcal{A} \cap \{t | t < T\}$. Following Garicano and Rayo (2017), I refer to a contract $\mathcal{C}$ with graduation T as a *delayed-reward* contract if the agent's liquidity constraint at period t_k binds for all $k = 1,2,\ldots,K$; and *a quasi-delayed-reward* contract, if the agent's liquidity constraint at period t_k binds for all $k = 1,2,\ldots,K-1$, but may not bind at $t_K = T-1$, that is, $\sum_{\tau=0}^{t_K}(1+r)^{t_K-\tau}w_\tau = \sum_{\tau=t_{K-1}}^{t_K}(1+r)^{t_K-\tau}w_\tau \geq 0$. Let $\mathcal{D}$ denote the set of delayed-reward contracts, and let $\mathcal{Q}$ denote the set of quasi-delayed-reward contracts (by definition, we have $\mathcal{D} \subset \mathcal{Q}$).

Step 1: Every optimal contract that solves problem (II) belongs to $\mathcal{Q}$. Let $\mathcal{C} = (y_t, w_t)_{t=0}^{\infty} \notin \mathcal{Q}$ be an optimal contract with a graduation date T and $y_T = f(1)$. Then, $\exists t^* \in \mathcal{A} \cap \{t | t < T-1\}$ such that $\sum_{\tau=0}^{t^*}(1+r)^{t^*-\tau}w_\tau > 0$. Consider a contract $\mathcal{C}' = (y_t', w_t') \in \mathcal{Q}$ with a graduation date S such that: (i) The agent's overall payoff is equal under $\mathcal{C}$ and $\mathcal{C}'$, i.e., $V_0(\mathcal{C}) = V_0(\mathcal{C}')$. This requires $\sum_{t=0}^{T-1}\delta^t w_t = \sum_{t=t_{K-1}+1}^{t_K=S-1}\delta^t w_t'$. (ii) For all knowledge-transfer periods $t \in \mathcal{A}$, the agent's incentive constraint binds, that is, $V_t(\mathcal{C}') = \frac{1}{1-\delta}y_t'$, $\forall t \in \mathcal{A}$.

Contract $\mathcal{C}'$ has the property that

$$V_{t+1}(\mathcal{C}') \geq V_{t+1}(\mathcal{C}), \forall t \in \mathcal{A}. \tag{A1}$$

For $t \in \mathcal{A}$ and $t < S$, (A1) follows from the property $V_0(\mathcal{C}) = V_0(\mathcal{C}')$ and the fact that $\sum_{\tau=0}^{t} \delta^\tau w_\tau' = 0$ and $\sum_{\tau=0}^{t} \delta^\tau w_\tau \geq 0$[4]. For $t \geq S$, (A1) follows from the fact that $V_{t+1}(\mathcal{C}') \geq \frac{1}{1-\delta} f(1) \geq V_{t+1}(\mathcal{C})$. Property (ii) and (A1) together imply that $y_t' \geq y_t$ for all $t \in \mathcal{A}$.

Moreover, we have

$$V_{t^*+1}(\mathcal{C}') > V_{t^*+1}(\mathcal{C}). \tag{A2}$$

When $T > t^*$, (A2) follows from property $V_0(\mathcal{C}') = V_0(\mathcal{C})$ and the fact that $\sum_{\tau=0}^{t^*} \delta^\tau w_\tau' = 0$ and $\sum_{\tau=0}^{t^*} \delta^\tau w_\tau > 0$. When $S \leq t^*$, (A2) follows from the fact that $V_{t^*+1}(\mathcal{C}') = \frac{1}{1-\delta} f(1) > V_{t^*+1}(\mathcal{C})$[5]. Property (ii) and (A2) imply that $y_{t^*}' > y_{t^*}$.

As a result, since $\mathcal{C}$ and $\mathcal{C}'$ deliver the same payoff for the agent at $t = 0$, we have

$$\Pi_0(\mathcal{C}') - \Pi_0(\mathcal{C}) = \sum_{t=0}^{\infty} \delta^t (y_t' - y_t) \geq \delta^{t^*}(y_t^* - y_t) > 0. \tag{A3}$$

Step 2: Every optimal contract $\mathcal{C}$ belongs to $\mathcal{D}$. Let $\mathcal{C} \in \mathcal{Q}$ be an optimal contract with a graduation date T. Since there is no knowledge between two knowledge-transfer periods t_k and t_{k+1}, the production is the same during this time range (i.e., $f(X_t) = y_{t_k+1}$ for all $t = t_k + 1, \ldots, t_{k+1}$). The expert's profits at $t = 0$ are

$$\Pi_0(\mathcal{C}) = \left(\sum_{k=1}^{K-1} y_{t_k+1} \sum_{\tau=t_k+1}^{t_{k+1}} \delta^\tau \right) - Z,$$

where $Z = \sum_{\tau=t_{K-1}+1}^{t_K} \delta^\tau w_\tau$ is the agent's cumulative wage during the last non-transfer periods $t_{K-1} < t \leq t_K = T - 1$. Binding the agent's incentive constraint at period $t_{K-1} + 1$ gives

$$V_{t_{K-1}+1}(\mathcal{C}) = \delta^{-(t_{K-1}+1)} Z + \frac{\delta^{T-(t_{K-1}+1)}}{1-\delta} f(1) = \frac{1}{1-\delta} y_{t_{K-1}+1}.$$

[4]For $t \in \mathcal{A}$ and $t < T$, we have $V_0(\mathcal{C}') = \delta^{t+1} V_{t+1}(\mathcal{C}') = \mathrm{V}_0(\mathcal{C}) = \sum_{\tau=0}^{t} \delta^\tau w_\tau + \delta^{t+1} V_{t+1}(\mathcal{C}) \geq \delta^{t+1} V_{t+1}(\mathcal{C})$

[5] For the inequality, note that $t^* < T - 1$ thus $f(X_{t^*+1}) < f(1)$, so that $\Pi_{t^*+1} + V_{t^*+1} < \frac{f(1)}{1-\delta}$. The expert's incentive constraint $\Pi_{t^*+1} \geq 0$ requires $V_{t^*+1} < \frac{f(1)}{1-\delta}$.

Binding the agent's incentive constraint at period $t_k + 1$ for $k = 1, \dots, K-2$, i.e., $V_{t_k+1}(\mathcal{C}) = \delta^{t_{K-1}-t_k} V_{t_{K-1}+1}(\mathcal{C}) = \frac{1}{1-\delta} y_k$, gives

$$y_{t_k+1} = (1-\delta)\delta^{-(t_k+1)}\left[Z + \frac{\delta^T}{1-\delta} f(1)\right], \forall t_k < t_{K-1}.$$

As a result, the expert's profits are a linear function of Z as

$$\left[\sum_{k=1}^{K-1}(1-\delta^{t_{k+1}-t_k}) - 1\right] Z + constant$$

Since the expert is free to choose $w_t \in [0, f(1)]$ for all $t = t_{K-1}+1, \dots, T-1$, the optimality of $\mathcal{C}$ requires $Z \in \left\{0, f(1)\sum_{\tau=t_{K-1}+1}^{t_K} \delta^\tau\right\}$. When $Z = 0$, the agent's liquidity constraint binds at period $T-1$. When $Z = f(1)\sum_{\tau=t_{K-1}+1}^{t_K} \delta^\tau$, which means $w_t = f(1)$, for all $t = t_{K-1}+1, \dots, T-1$, the agent's graduation date becomes $T = t_{K-1}+1$. In both cases, the agent's liquidity constraint binds at all knowledge-transfer periods before graduation, i.e., $\mathcal{C} \in \mathcal{D}$. ■

Proof of Proposition 2:

Let $\mathcal{C}$ be a feasible contract (satisfying (1)-(3)) which has a graduation date T and satisfies the properties in Proposition 1. Since the agent's liquidity constraint binds at all t_k, the expert's profits at $t = 0$ can be written as

$$\sum_{K=1}^{K-1} y_{t_k+1} \sum_{\tau=t_k+1}^{t_{k+1}} \delta^\tau.$$

The agent's incentive constraint at period $t = t_k + 1$ is

$$\frac{\delta^{T-(t_k+1)}}{1-\delta} f(1) \geq \frac{1}{1-\delta} y_{t_k+1}.$$

Binding the agent's incentive constraint at $t = t_k + 1$ gives $y_{t_k+1} = \delta^{T-(t_k+1)} f(1)$, that is, $f(X_{t+1}) = \delta^{T-(t+1)} f(1), \forall t \in \mathcal{A} \cap \{t | t < T\}$. ■

Proof of Lemma 3:

Step 1: For every optimal contract $\mathcal{C} \in \mathcal{H}$ with the maximum graduation date $T_{max} = T_i$, $i \in \{1,2\}$, $T_i \geq i - \frac{1}{\ln \delta}$. Suppose $T_i < i - \frac{1}{\ln \delta}$, the expert can increase her profits by increasing T_i to $T_i' = i - \frac{1}{\ln \delta}$, and train agent i during periods $T_i \leq t < T_i'$. To see this, consider another contract $\mathcal{C}' \in \mathcal{H}$ in which $T_i' = T_i^*$ and $T_j' = T_j$ for $j \neq i$. Since $t_{i,1} = i - 1$ for $i \in \{1,2\}$, the expert's profits obtained from agent i in $\mathcal{C}'$ relative to $\mathcal{C}$ is

$$\frac{\Pi_{i,0}(\mathcal{C}')}{\Pi_{i,0}(\mathcal{C})} = \delta^{T_i' - T_i}\left[1 + \frac{(T_i' - T_i)(1-\delta)}{\sum_{k=1}^{K_i - 1}(1 - \delta^{t_{i,k+1} - t_{i,k}})}\right]$$

$$> \delta^{T_i' - T_i}\left[1 + \frac{(T_i' - T_i)(1-\delta)}{(T_i - i)(1-\delta)}\right]$$

$$= \frac{\delta^{T_i' - i}(T_i' - i)}{\delta^{T_i - i}(T_i - i)} > 1.$$

The first inequality follows that the nominator is smallest when the knowledge transfer is consecutive before T_i. The last inequality follows from the fact that $\arg\max_x \delta^{x-i}(x - i) = i - \frac{1}{\ln \delta}$.

Step 2: In every optimal contract $\mathcal{C} \in \mathcal{H}$, agent 1 graduates before agent 2 and agent 1 is trained during the m consecutive knowledge-transfer periods after alternate training. Consider a contract $\mathcal{C} \in \mathcal{H}$ that satisfies the property in step 1. There are four cases, depending on whether agent 1 or 2 is trained during the m consecutive transfer periods at $2n \leq t < T_i$, and whether agent 1 or 2 graduates first.

When $T_2 > T_1$, the profits when agent 1 is trained during the m consecutive knowledge-transfer periods are lower than that when agent 2 is trained during these periods[6]. When $T_1 > T_2$, if agent 2 is trained during the m periods, the profits are lower than those in a contract $\mathcal{C}' \in \mathcal{H}$ with $T_1' = T_2$ and $T_2' = T_1$, in which agent 1 is trained during the m consecutive periods, and agent 2 is trained during the l

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[6] The profit differences between the two cases are $\delta^{T_1}\left(m + 1 + \delta - \frac{1-\delta^{m+2}}{1-\delta}\right)f(1) - \delta^{T_2}\left(m - \delta^2\frac{1-\delta^m}{1-\delta}\right)f(1) \geq \delta^{T_1}\left[m + 1 + \delta - \frac{1-\delta^{m+2}}{1-\delta} - \left(m - \delta^2\frac{1-\delta^m}{1-\delta}\right)\right]f(1) = 0.$

periods after $t = T_1' - 1$[7]. If agent 1 is trained during the m periods in $\mathcal{C}$, it can be shown that the optimal m is either 0 or 1[8]. Suppose $\mathcal{C}$ is optimal, consider a contract $\mathcal{C}'$ with $T_1' = T_2$ and $T_2' = T_1$ (thus $T_2' > T_1'$), in which agent 2 is trained during the m periods. The profits difference between $\mathcal{C}'$and $\mathcal{C}$ is $\delta^{T_2+1}(\delta^m - \delta^{T_1-T_2})f(1)$. Since in optimal contracts $\mathcal{C}$, $m \in \{0,1\}$, $\Pi_0(\mathcal{C}') \geq \Pi_0(\mathcal{C})$. Also, we have shown that the expert's profits will be higher by training agent 1 during the m periods in $\mathcal{C}'$, so $\mathcal{C}$ is not optimal, a contradiction.

Step 3: In every optimal contract $\mathcal{C} \in \mathcal{H}$, the expert alternates knowledge transfer between the two agents as much as possible, i.e., $m \in \{0,1\}$. Consider a contract $\mathcal{C} \in \mathcal{H}$ that satisfies the property in step 2. The expert's profits are

$$\delta^{T_1}[n(1+\delta)+m]f(1) + \delta^{T_2}\left[n(1+\delta) + \delta^2\frac{1-\delta^m}{1-\delta} + l\right]f(1).$$

Substituting $n = \frac{T_1-m-1}{2}$, and $l = T_2 - T_1 - 1$ into the above expression, the second-order derivative of $\Pi_0(\mathcal{C})$ with respect to m is $-\frac{\delta^{2+m+T_2}(\ln\delta)^2}{1-\delta}f(1) < 0$. Also, given fixed T_1, T_2, the profit difference between contracts with $m = m$ and $m = m+1$ is

$$\frac{1}{2}\delta^{T_1}[\delta^{T_2-T_1}(1+\delta-2\delta^{2+m}) - (1-\delta)] > 0,$$

given that $1 - \frac{1}{\ln\delta} \geq 4$ and $T_2 - T_1 \geq 1$. Note that the minimum value of m is 0 when T_1 is odd, and 1 when T_1 is even. Therefore, at optimality, $m \in \{0,1\}$.

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[7] The overall profits given by $\mathcal{C}$ are

$$\delta^{T_1}\left[n(1+\delta) + \delta^2\frac{1-\delta^{m+1}}{1-\delta} + l\right]f(1) + \delta^{T_2}[(n-1)(1+\delta)+m+1]f(1),$$

where the first and second terms are profits obtained from agent 1 and 2, respectively. The profit difference between $\mathcal{C}'$ and $\mathcal{C}$ is $\delta(\delta^{T_2} - \delta^{T_1}\delta^{m+1})f(1) > 0$.

[8] The overall profits in this case are

$$\Pi_0(\mathcal{C}) = \delta^{T_1}[n(1+\delta)+m+\delta+l]f(1) + \delta^{T_2}\left[(n-1)(1+\delta) + \frac{1-\delta^{m+1}}{1-\delta}\right]f(1)\,.$$

Since $n = \frac{T_2-m-1}{2}$, the first-order and second-order derivatives are

$$\frac{\partial\Pi_0(\mathcal{C})}{\partial m}\Big|_{m=1} = \frac{(1-\delta)^2\delta^{T_1} - [(1-\delta^2) + 2\delta^2\ln\delta]\delta^{T_2}}{2(1-\delta)}f(1) \leq \frac{\partial\Pi_0(\mathcal{C})}{\partial m}\Big|_{m=1, T_1=1-\frac{1}{\ln\delta}, T_2=-\frac{1}{\ln\delta}} < 0,$$

$$\frac{\partial^2\Pi_0(\mathcal{C})}{\partial m^2} = -\frac{\delta^{1+m+T_2}(\ln\delta)^2}{1-\delta}f(1) < 0.$$

Step 4: In every optimal contract $\mathcal{C} \in \mathcal{H}$, $T_2 = T_1 + 1$ (i.e., $l = 0$). Consider a contract $\mathcal{C} \in \mathcal{H}$ that satisfies the properties in steps 1, 2 and 3. When $m = 0$, since $T_1 = 2n + 1$, $T_2 = T_1 + l + 1$, the profits are a function of n and l as

$$\Pi_0(n, l) = \delta^{2n+1} n(1+\delta) f(1) + \delta^{2(n+1)+l}[n(1+\delta) + l] f(1).$$

When $T_2 > T_1 \geq T^*$, if $l > 0$, the expert can increase $\Pi_{2,0}$ by reducing T_2 to $T_1 + 1$[9], so that $l = 0$, and this operation does not change $\Pi_{1,0}$. When $T_1 < T^* < T_2$, the expert can increase her profits by reducing T_2 to $T^* + 1$[10]. Therefore, consider $T_1 < T_2 = T^* + 1$, the expert's profits change with l as

$$\Pi_0(l) = \delta^{T_2}\left[\left(1 + \delta^{-l-1}\right)\frac{T_2 - 2 - l}{2}(1+\delta) + l\right] f(1).$$

Since $\frac{\partial^2 \Pi_0(l)}{\partial l^2} < 0$, $\frac{\partial \Pi_0(l)}{\partial l}|_{l=0} > 0$, and $\frac{\partial \Pi_0(l)}{\partial l}|_{l=1} < 0$[11], given $\frac{\Pi_0(l=0)}{\Pi_0(l=1)} > 1$ when $T_2 \geq 3$, $l = 0$ is optimal.

When $m = 1$, the overall profits change with n and l as

$$\Pi_0(n, l) = \delta^{2n+2}(n(1+\delta) + 1) + \delta^{2n+3+l}(n(1+\delta) + \delta^2 + l) f(1).$$

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[9] Let $\mathcal{C}' \in \mathcal{H}$ be the contract after reducing T_2 to $T_1 + 1$, the profits obtained from agent 2 in $\mathcal{C}$ relative to $\mathcal{C}'$ are

$$\frac{\Pi_{2,0}(\mathcal{C})}{\Pi_{2,0}(\mathcal{C}')} = \delta^l \left[1 + \frac{l}{n(1+\delta)}\right],$$

The first-order derivative of $\frac{\Pi_{2,0}(\mathcal{C})}{\Pi_{2,0}(\mathcal{C}')}$ with respect to l is

$$\delta^l \frac{1 + (l + (1+\delta)n)\ln\delta}{n(1+\delta)} \leq \delta^l \left(\frac{1}{n(1+\delta)} + \ln\delta\right) \leq \delta^l \left(\frac{1}{\frac{T^*(1+\delta)}{2}} + \ln\delta\right) < 0.$$

[10] The profits is a function of T_1 and T_2 as $\Pi_0(\mathcal{C}) = \delta^{T_1}\frac{T_1-1}{2}(1+\delta) + \delta^{T_2}\left[\frac{T_1-1}{2}(1+\delta) + T_2 - T_1 - 1\right]$. The first-order derivative w.r.t. T_2 is $\delta^{T_2}\left[1 + \frac{1}{2}(2T_2 - T_1(1-\delta) - \delta - 3)\ln\delta\right]$. Given $T_1 < T^*$ and $T^* = 1 - \frac{1}{\ln\delta} \geq 3$, $\frac{\partial \Pi_0}{\partial T_2} < 0$ when $T_2 > 1 + T^*$. The profit difference between $T_2 = 1 + T^*$ and $T = 2 + T^*$ are

$$\delta^2\left[\frac{1}{2e}(1 - T_1(1-\delta)^2 - (4-\delta)\delta) - \frac{1-\delta}{eln\,\delta}\right] \geq \delta^2\left[\frac{1}{2e}(1 - (T^*-1)(1-\delta)^2 - (4-\delta)\delta) - \frac{1-\delta}{eln\,\delta}\right] \geq 0$$

[11] Note that

$$\frac{\partial^2 \Pi_0(l)}{\partial l^2} = \frac{\delta^{1-l}(1+\delta)\ln\delta\,(1 - l\ln\delta)}{2e} f(1) < 0,$$

$$\frac{\partial \Pi_0(l)}{\partial l}|_{l=0} = \frac{(1-\delta)\delta^2}{2e} f(1) > 0.$$

$$\frac{\partial \Pi_0(l)}{\partial l}|_{l=1} = \frac{(1-\delta)\delta^2 + (1+\delta)\ln\delta}{2e} f(1) < 0.$$

When $T_2 > T_1 \geq T^*$, if $l > 0$, similarly, the expert can increase $\Pi_{2,0}$ by reducing l to 0[12]. When $T_1 < T^* < T_2$, the expert can increase her profits by reducing T_2 to $T^* + 1$. When $T_1 < T_2 = T^* + 1$, the profits change with l as

$$\Pi_0(l) = \delta^{T_2}\left[\left(1 + \delta^{-l-1}\right)\frac{T_2 - 3 - l}{2}(1 + \delta) + l + \delta^2 + \delta^{-1-l}\right] f(1).$$

Similarly, $\frac{\partial^2 \Pi_0(l)}{\partial l^2} < 0$, $\frac{\partial \Pi_0(l)}{\partial l}|_{l=0} > 0$, and $\frac{\partial \Pi_0(l)}{\partial l}|_{l=1} < 0$[13], given $\frac{\Pi_0(l=0)}{\Pi_0(l=1)} > 1$ when $T_2 \geq 3$, $l = 0$ is optimal.

Step 5: Up to an integer constraint of $T^* = 1 - \frac{1}{\ln \delta}$, in every optimal contract $\mathcal{C} \in \mathcal{H}$, $T_1 = T^* = 1 - \frac{1}{\ln \delta}$, $T_2 = T^* + 1$. Consider a contract $\mathcal{C} \in \mathcal{H}$ that satisfies the properties in steps 1,2,3 and 4. When $m = 0$, which occurs when T^* is odd, the corresponding profits are

$$\delta^{T_1}(1 + \delta)^2 \frac{T_1 - 1}{2} f(1).$$

The first-order condition gives that at the optimality, $T_1 = T^* = 1 - \frac{1}{\ln \delta}$. When $m = 1$, which occurs when T^* is even, the corresponding profits are

$$\delta^{T_1}\left[(1 + \delta)^2\left(\frac{T_1}{2} - 1\right) + 1 + \delta^3\right] f(1).$$

At optimality, $T_1 = T^*$[14]. ■

[12] Note that the ratio of the profits obtained from agent 2 in $\mathcal{C}$ relative to $\mathcal{C}'$, and its first-derivative w.r.t. l are:

$$\frac{\Pi_{2,0}(\mathcal{C})}{\Pi_{2,0}(\mathcal{C}')} = \delta^l \frac{l + n(1 + \delta) + \delta^2}{n(1 + \delta) + \delta^2},$$

$$\frac{\partial\left(\frac{\Pi_{2,0}(\mathcal{C})}{\Pi_{2,0}(\mathcal{C}')}\right)}{\partial l} = \delta^l \frac{1 + (l + n(1 + \delta) + \delta^2)\ln \delta}{n(1 + \delta) + \delta^2} \leq \delta^l\left(\frac{1}{\frac{T^*(1 + \delta)}{2} + \delta^2} + \ln \delta\right) < 0.$$

Therefore, $\frac{\Pi_{2,0}(\mathcal{C})}{\Pi_{2,0}(\mathcal{C}')} \leq 1$.

[13] Note that given $T_2 = 2 - \frac{1}{\ln \delta} \geq 3$,

$$\frac{\partial^2 \Pi_0(l)}{\partial l^2} = \frac{\delta^{1-l} \ln \delta \left(1 + \delta - \ln \delta\,(l - 1 + \delta + l\delta)\right)}{2e} f(1) < 0,$$

$$\frac{\partial \Pi_0(l)}{\partial l}|_{l=0} = \frac{\delta(1 - \delta)(\delta - \ln \delta)}{2e} f(1) > 0,$$

$$\frac{\partial \Pi_0(l)}{\partial l}|_{l=1} = \frac{\delta(\delta - \delta^2 + 2\ln \delta)}{2e} f(1) < 0.$$

[14] The first-order condition gives

$$T_1 = \frac{2(2 - \delta)\delta}{1 + \delta} - \frac{1}{\ln \delta} = T^* + \left(5 - 2\delta - \frac{6}{1 + \delta}\right).$$

Proof of Proposition 3:

When $\delta \geq e^{-1/2}$ so that $T^* = 1 - \frac{1}{\ln \delta} \geq 3$, based on Proposition 2 and Lemmas 1 and 3, under the integer constraint of $T^* = 1 - \frac{1}{\ln \delta}$ and $\frac{\delta}{e \ln \delta}$, the expert's maximum profits under different scenarios are as follows:

(i) Agent 2 is not trained before agent 1 graduates, $\Pi_0^1 = e^{-1+\frac{\delta}{e}} \delta (T^* - 1) f(1)$.

(ii) Agent 2 is trained before agent 1 graduates and T^* is odd, $\Pi_0^2 = \delta^{T^*} (1+\delta)^{2\frac{T^*-1}{2}} f(1)$.

(iii) Agent 2 is trained before agent 1 graduates and T^* is even, $\Pi_0^3 = \delta^{T^*} \left[(1+\delta)^2 \left(\frac{T^*}{2} - 1 \right) + 1 + \delta^3 \right] f(1)$.

Ignoring the integer constraints, it can be shown that $\Pi_0^2 > \Pi_0^1$ and $\Pi_0^3 > \Pi_0^1$ given $T^* \geq 3$[15].

When $\delta < e^{-\frac{1}{2}}$, the optimal contract when agent 2 is trained before agent 1 is alternate training with $T_1 = 3$ and $T_2 = 4$. We still need to solve for the optimal contract when agent 2 is not trained before agent 1 graduates. First consider agent 2's graduation date, the agent 2's optimal training duration is 3 periods when $0.5 < \delta < e^{-\frac{1}{2}}$ and 2 periods when $\delta < 0.5$[16]. When $0.5 < \delta < e^{-\frac{1}{2}}$ so that agent 2's training duration is 3 periods, Lemma 1 indicates that the optimal graduation date of agent 1 is no larger than 3 periods. When $T_1 = 3$, the expert's profits are lower than the contract with alternate training[17].

Since $-1 < 5 - 2\delta - \frac{6}{1+\delta} < 1$, we need to compare the profits $\Pi_0(T_1)$ at $T_1 = T^* - 1$, T^* and $T^* + 1$. The profit difference when $T_1 = T^*$ and $T_1 = T^* + 1$ is

$$-\frac{\delta(1+\delta)\left[1 - \delta^2 - \ln \delta \left(1 - \delta\left(5 - 2\delta(2-\delta)\right)\right)\right]}{2e \ln \delta} f(1) > 0.$$

Also, given $T^* = 1 - \frac{1}{\ln \delta} \geq 3$, the profit difference when $T_1 = T^*$ and $T_1 = T^* - 1$ is

$$\frac{(1+\delta)\left[1 - \delta^2 + \delta \ln \delta \left(5 - \delta(5 - 2\delta)\right)\right]}{2e \ln \delta} f(1) > 0.$$

[15]When $T^* = 1 - \frac{1}{\ln \delta}$ is odd, note that $\frac{\Pi_0^2}{\Pi_0^1} = \frac{1}{2} e^{-\frac{\delta}{e}} (1+\delta)^2$ is increasing in δ, and $\Pi_0^2 > \Pi_0^1$ when $T^* = 3$. When T^* is even and $T^* \geq 4$, $\frac{\Pi_0^3}{\Pi_0^1} = \frac{1}{2} e^{-\frac{\delta}{e}} (1+\delta)\left[1 + \delta - \ln \delta \left(1 - \delta(3 - 2\delta)\right)\right]$ is increasing in δ, and $\Pi_0^3 > \Pi_0^1 1$ when $T^* = 4$.

[16] Consider a single-agent contract, the expert's profits are $\delta^2 f(1)$ and $2\delta^3 f(1)$ when the graduate date is 2 and 3, respectively. $\delta^2 f(1) > 2\delta^3$ when $\delta < 0.5$.

[17]The expert's profits are $2\delta^3(1+\delta^3) f(1)$ and $\delta^3(1+\delta)^2 f(1)$ under sequential training with $T_1 = 3$ and $T_2 = 6$, and alternate training with $T_1 = 3$, $T_2 = 4$, respectively. $2\delta^3(1+\delta^3) f(1) > \delta^3(1+\delta)^2 f(1)$ requires $\delta < 0.5$.

When $T_1 = 2$, this contract gives higher profits than the contract with alternate training when $0.5 < \delta < \delta^{*}$[18], where $\delta^{*} \approx 0.555$ solves the equation $\delta^3 - 2\delta^2 - \delta + 1 = 0$. When $\delta < 0.5$, agent 2's training duration is 2 periods, at optimality, agent 1's optimal graduation date is 2, and this contract gives higher profits than the contract with alternate training.

To relax the integer constraint of $1 - \frac{1}{\ln \delta}$, let $\Pi_0^2(T) = \delta^T (1+\delta)^2 \frac{T-1}{2} f(1)$ and $\Pi_0^3(T) = \delta^T \left[(1+\delta)^2 \left(\frac{T}{2} - 1\right) + 1 + \delta^3\right] f(1)$ denote the profits when T is odd and even, respectively. When $\underline{T}$ is even, we need to compare the profits under four cases: $\Pi_0^2(\underline{T}-1)$, $\Pi_0^2(\underline{T}+1)$, $\Pi_0^3(\underline{T})$, $\Pi_0^3(\underline{T}+2)$. It can be shown that $\Pi_0^2(\underline{T}+1) > \Pi_0^2(\underline{T}-1)$[19] and $\Pi_0^2(\underline{T}+1) > \Pi_0^3(\underline{T}+2)$[20], so we only need to compare $\Pi_0^2(\underline{T}+1)$ and $\Pi_0^3(\underline{T})$. It turns out that $\Pi_0^3(\underline{T}) > \Pi_0^2(\underline{T}+1)$ when $\underline{T} > \frac{2\delta(2-\delta)}{1-\delta^2}$[21].

When $\underline{T}$ is odd, we compare the profits under four cases: $\Pi_0^2(\underline{T})$, $\Pi_0^2(\underline{T}+2)$, $\Pi_0^3(\underline{T}-1)$, $\Pi_0^3(\underline{T}+1)$. It can be shown that $\Pi_0^3(\underline{T}+1) > \Pi_0^3(\underline{T}-1)$[22] and $\Pi_0^3(\underline{T}+1) > \Pi_0^2(\underline{T}+2)$[23]. Therefore, we only need to compare profits $\Pi_0^2(\underline{T})$ and $\Pi_0^3(\underline{T}+1)$, and $\Pi_0^2(\underline{T}) > \Pi_0^3(\underline{T}+1)$ when $\underline{T} > \frac{1+\delta(2+2\delta^2-3\delta)}{1-\delta^2}$[24]. ■

Proof of Corollary 2:

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[18]Under sequential training with $T_1 = 2$ and $T_2 = 5$, and alternate training with $T_1 = 3$, $T_2 = 4$, the expert's profits are $\delta^2(1+2\delta^3)f(1)$ and $\delta^3(1+\delta)^2 f(1)$, respectively. $\delta^2(1+2\delta^3)f(1) > \delta^3(1+\delta)^2 f(1)$ when $\delta^3 - 2\delta^2 - \delta + 1 > 0$.

[19]Note that $\Pi_0^2(T+1) - \Pi_0^2(T-1) = \frac{1}{2}\delta^{T-1}(1+\delta)^2[2 - T(1-\delta^2)]f(1)$. Therefore, $\Pi_0^2(T+1) > \Pi_0^2(T-1)$ when $T < \frac{2}{1-\delta^2}$. Since $\frac{2}{1-\delta^2} > \frac{1}{1-\ln\delta} > \underline{T}$, $\Pi_0^2(\underline{T}+1) > \Pi_0^2(\underline{T}-1)$.

[20]Note that $\Pi_0^2(T+1) - \Pi_0^3(T+2) = -\frac{1}{2}\delta^{T+1}(1+\delta)\left[2\delta\left(1-\delta(1-\delta)\right) - T(1-\delta^2)\right]f(1)$. Therefore, $\Pi_0^2(T+1) > \Pi_0^3(T+2)$ when $T > \frac{2\delta(1-\delta+\delta^2)}{1-\delta^2}$. Since $\frac{2\delta(1-\delta+\delta^2)}{1-\delta^2} < \underline{T}$ for all $\delta \in (0,1)$, $\Pi_0^2(\underline{T}+1) > \Pi_0^3(\underline{T}+2)$.

[21]Note that $\Pi_0^3(T) - \Pi_0^2(T+1) = -\frac{1}{2}\delta^T(1+\delta)[4\delta - 2\delta^2 - (1-\delta^2)T]f(1)$.

[22]Note that $\Pi_0^3(T+1) - \Pi_0^3(T-1) = \frac{1}{2}\delta^{T-1}(1+\delta^2)\left[1 - T(1-\delta^2) + \delta\left(4 - \delta(5-2\delta)\right)\right]f(1)$, so $\Pi_0^3(T+1) > \Pi_0^3(T-1)$ when $T < 5\left(1 - \frac{1}{1+\delta}\right) + \frac{1}{1-\delta} - 2\delta$. Since $5\left(1 - \frac{1}{1+\delta}\right) + \frac{1}{1-\delta} - 2\delta > \frac{1}{1-\ln\delta} > \underline{T}$ for $1 - \frac{1}{\ln\delta} > 3$, $\Pi_0^3(\underline{T}+1) > \Pi_0^3(\underline{T}-1)$.

[23] Note that $\Pi_0^3(T+1) - \Pi_0^2(T+2) = -\frac{1}{2}\delta^{T+1}(1+\delta)[4\delta - \delta^2 - 1 - (1-\delta^2)T]f(1)$, so $\Pi_0^3(T+1) > \Pi_0^2(T+2)$ when $T > \frac{4\delta-\delta^2-1}{1-\delta^2}$. Since $\frac{4\delta-\delta^2-1}{1-\delta^2} < T - 1 < \underline{T}$, $\Pi_0^3(\underline{T}+1) - \Pi_0^2(\underline{T}+2)$.

[24]Note that $\Pi_0^2(T) - \Pi_0^3(T+1) = -\frac{1}{2}\delta^T(1+\delta)\left[1 - T(1-\delta^2) + \delta^2\left(2 - \delta(3-2\delta)\right)\right]f(1)$.

Part (*i*)*:* Consider a contract with a single agent constraint of non-transfer periods $\mathcal{B}$. The Planner's problem is to maximize the objective $\lambda V_0(\mathcal{C}) + \Pi_0(\mathcal{C})$ subject to the constraints in Problem (II). The goal is to show that the optimal contract $\mathcal{C}$ belongs to the set of *delayed-reward* contracts $\mathcal{D}$. Step 1 in the proof of property (ii) in Proposition 1 implies that $\mathcal{C}$ must belong to the set of quasi-delayed-reward contracts $\mathcal{Q}$. Given $\mathcal{C} \in \mathcal{Q}$, there exists a graduation date $T \geq 1$ such that the agent's liquidity constraint binds at all knowledge-transfer periods before T (i.e., $t \in \mathcal{A} \cap \{t | t < T - 1\}$, $Z = \sum_{\tau=t_{K-1}+1}^{t_K} \delta^\tau w_\tau \geq 0$, where t_{K-1} and $t_K = T - 1$ are the last two knowledge-transfer periods, and $w_t = f(1)$ for all $t > T$.

The Planner's objective is

$$\lambda V_0(\mathcal{C}) + \Pi_0(\mathcal{C}) = \lambda \left[Z + \frac{\delta^T}{1-\delta} f(1) \right] + \sum_{k=1}^{K-1} y_{t_k+1} \sum_{\tau=t_k+1}^{t_{k+1}} \delta^\tau - Z.$$

The agent's incentive constrains after knowledge-transfer periods $t_1, \ldots, t_{K-1}$ are

$$V_{t_k+1}(\mathcal{C}) = \delta^{t_{K-1}-t_k} V_{t_{K-1}+1}(\mathcal{C}) \geq \frac{1}{1-\delta} y_{t_k+1},$$

where $V_{t_{K-1}+1}(\mathcal{C}) = \delta^{-(t_{K-1}+1)} \left[Z + \frac{\delta^T}{1-\delta} f(1) \right]$. Since the Planner's objective is increasing in $y_{t_1+1}, \ldots, y_{t_{K-1}+1}$, the hypothesis that $\mathcal{C}$ maximizes the Planner's object requires that the incentive constraints above bind. After substituting for y_{t_k+1}, the Planner's objective becomes

$$\left(\sum_{k=1}^{K-1} (1 - \delta^{t_{k+1}-t_k}) + \lambda - 1 \right) Z + constant,$$

which is linear in Z. Since the expert is free to vary Z in the range $\left[0, f(1) \sum_{\tau=t_{K-1}+1}^{t_K} \delta^\tau \right]$, the hypothesis that $\mathcal{C}$ maximizes the Planner's object requires that $Z \in \left\{0, f(1) \sum_{\tau=t_{K-1}+1}^{t_K} \delta^\tau \right\}$. As a result, for both cases, $\mathcal{C} \in \mathcal{D}$.

Finally, given a fixed graduation date T, since the agent's payoff is fixed, maximizing the Planner's objective is effectively the same as maximizing the expert's profits. Therefore, the optimal knowledge stock is the same as that described in Proposition 2.

Part (ii): There are two scenarios depending on whether agent 2 gets trained before agent 1 graduates or not. For the case when agent 2 gets trained before agent 1 graduates, it is sufficient to focus on contracts in the set $\mathcal{H}$, which have the special properties described in Lemma 2, since in the proof of Lemma 2, the agents' graduation date is fixed, so maximization of the Planner's objective is effectively the same as maximization of the expert's profits. Now I show that a Pareto-efficient contract in which agent 2 is trained before agent 1's graduation preserves the overall structure of profit-maximizing contract.

Step 1: For every efficient contract $\mathcal{C} \in \mathcal{H}$, the maximum graduation date is no smaller than $i - \frac{1}{\ln \delta} - \frac{2\lambda}{1-\delta^2}$, where $i \in \{1,2\}$ stands for the agent who graduates later. Denote the welfare from contracting with agent i as $W_{i,0}(\mathcal{C}) = \Pi_{i,0}(\mathcal{C}) + \lambda V_{i,0}(\mathcal{C})$. If $T_i < i - \frac{1}{\ln \delta}$, the Planner can increase W_0 by increasing l so that in the new contract $\mathcal{C}' \in \mathcal{H}$, the graduation date of agent i is $i - \frac{1}{\ln \delta}\max\{1 - \lambda A, 0\}$. The relative value of $W_{i,0}$ in $\mathcal{C}'$ and $\mathcal{C}$ is

$$\frac{W_{i,0}(\mathcal{C}')}{W_{i,0}(\mathcal{C})} = \delta^{T_i' - T_i}\left[1 + \frac{(T_i' - T_i)(1-\delta)}{\sum_{k=1}^{K_i - 1}(1 - \delta^{t_{i,k+1} - t_{i,k}}) + \lambda}\right]$$

$$> \delta^{T_i' - T_i}\left[1 + \frac{(T_i' - T_i)(1-\delta)}{(T_i - i)(1-\delta) + \lambda}\right]$$

$$= \frac{\delta^{T_i'}\left[\frac{(T_i' - i)}{2}(1+\delta) + \frac{\lambda}{1-\delta}\right]}{\delta^{T_i}\left[\frac{(T_i - i)}{2}(1+\delta) + \frac{\lambda}{1-\delta}\right]} > 1.$$

The last inequality follows from the fact that $\arg\max_{x \geq 1} \delta^x \left(\frac{x-i}{2}(1+\delta) + \frac{\lambda}{1-\delta}\right) = i - \frac{1}{\ln \delta} - \frac{2\lambda}{1-\delta^2}$.

Step 2: In every efficient contract $\mathcal{C} \in \mathcal{H}$, $T_2 > T_1$ and $m \in \{0,1\}$, and agent 1 is trained during period $t = 2n$ when $m = 1$. Suppose in an efficient contract $\mathcal{C} \in \mathcal{H}$, $T_1 > T_2$, step 2 in the proof of Lemma 2 shows that there exists $\mathcal{C}' \in \mathcal{H}$ with $T_i' \in \{T_1, T_2\}$ for $i = 1, 2$, such that $\Pi_0(\mathcal{C}') > \Pi_0(\mathcal{C})$. Also, the agents' total payoffs are not changed, since $V_{1,0}(\mathcal{C}') + V_{2,0}(\mathcal{C}') = \frac{\delta^{T_1'} + \delta^{T_2'}}{1-\delta} f(1) = \frac{\delta^{T_1} + \delta^{T_2}}{1-\delta} f(1)$.

Therefore, the Planner's objectives are higher in $\mathcal{C}'$ than $\mathcal{C}$, a contradiction. Now consider $\mathcal{C} \in \mathcal{H}$ with

$T_1 < T_2$. Given T_1 and T_2 fixed, the agents' profits are fixed, steps 3 in the proof of Lemma 2 show that the expert's profits are highest when $m \in \{0,1\}$. Moreover, if $m = 1$, step 2 in the proof of Lemma 2 shows that the expert's profits are higher when agent 1 (instead of agent 2) is trained in period $t = 2n$.

Step 3: In every efficient contract $\mathcal{C} \in \mathcal{H}$, $T_2 = T_1 + 1$. Consider a contract $\mathcal{C} \in \mathcal{H}$ that satisfies the properties in steps 1 and 2 (thus $T_2 > T_1$). Let $T^* = 1 - \frac{1}{\ln \delta} - \frac{2\lambda}{1-\delta^2}$. When $m = 0$, if $T_1 \geq T^*$, if $l > 0$, the planner can increase $W_{2,0}$ by reducing T_2 to $T_1 + 1$ (so that $l = 0$), and this operation does not change $W_{1,0}$[25]. When $T_1 \leq T^* < T_2$, the Planner can increase $W_{2,0}$ by reducing T_2 to $T^* + 1$. Therefore, consider $T_1 < T_2 = 1 + T^*$, the expert's profits change with l as

$$W_0(l) = \delta^{T_2}\left[\lambda \frac{\delta^{-l-1} + 1}{1-\delta} + \left(1 + \delta^{-l-1}\right)\frac{T_2 - 2 - l}{2}(1+\delta) + l\right] f(1).$$

Since $\frac{\partial^2 W_0(l)}{\partial l^2} < 0$ and $\frac{\partial \mathrm{W}_0(l)}{\partial l}|_{l=1} < 0$[26], given $\frac{\mathrm{W}_0(l=0)}{W_0(l=1)} > 1$ when $T_2 = T^* + 1 \geq 3$, $l = 0$ is optimal. When $m = 1$, a similar proof shows that $l = 0$ is optimal.

Step 4: Let $T_\lambda^* = 1 - \frac{1}{1-\ln \delta} - \frac{2\lambda}{1-\delta^2}$, up to an integer constraint of T_λ^* with $T_\lambda^* \geq 3$, in an optimal contract that agent the optimal graduation date is $T_1 = T_\lambda^*$ and $T_2 = T_\lambda^* + 1$**.** Consider a contract $\mathcal{C}$ that satisfies the properties in part (i) and the previous steps 1 to 4. When T_1 odd, so that fully alternate training is achievable, the Planner's objective is

$$\left[\lambda \frac{\delta^{T_1} + \delta^{T_1+1}}{1-\delta} + \delta^{T_1}(1+\delta)^2 \frac{T_1 - 1}{2}\right] f(1).$$

The first-order condition gives that the optimal T_1 is $1 - \frac{1}{\ln \delta} - \frac{2\lambda}{1-\delta^2}$. When T_1 is even, the Planner's objective is

[25] $W_{2,0}(n,l) = \left[\lambda \frac{\delta^{2(n+1)+l}}{1-\delta} + \delta^{2(n+1)+l}(n(1+\delta)+l)\right] f(1)$. The first-derivate of $W_{2,0}(n,l)/W_{2,0}(n,0)$ w.r.t. l is

$$\frac{\delta^l\left[1 - \delta + \left((1-\delta)\left(l + n(1+\delta)\right) + \lambda\right)\ln \delta\right]}{n(1-\delta^2)+\lambda} \leq \frac{\delta^l}{n(1-\delta^2)+\lambda}\left[1 - \delta + \left((1-\delta)\frac{T^*}{2}(1+\delta) + \lambda\right)\ln \delta\right] < 0.$$

[26] Note that $\frac{\partial^2 W_0(l)}{\partial l^2} = \frac{\delta^{1-l-\frac{2\lambda}{1-\delta}}(1+\delta)\ln \delta(1 - l\ln \delta)}{2e} f(1) < 0$ and $\frac{\partial \Pi_0(l)}{\partial l}|_{l=1} = \delta^{-\frac{2\lambda}{1-\delta^2}} \frac{(1-\delta)\delta^2 + (1+\delta)\ln \delta}{2e} f(1) < 0$.

$$\left[\lambda\frac{\delta^{T_1}+\delta^{T_1+1}}{1-\delta}+\delta^{T_1}\left((1+\delta)^2\left(\frac{T_1}{2}-1\right)+1+\delta^3\right)\right]f(1).$$

It can be shown that given T_λ^* is integer, the optimal T_1 is also T_λ^*[27].

Step 6: Up to an integer constraint of $T_\lambda^*=1-\frac{1}{\ln\delta}-\frac{2\lambda}{1-\delta^2}$ with $T_\lambda^*\geq 3$, the maximum value of the Planner's objective when agent 2 is trained before agent 1's graduation is higher than that when agent 2 is trained after agent 1's graduation. First, I derive the maximum value of the objective when agent 2 gets trained after agent 1's graduation. By Corollary 1 in Garicano and Rayo (2017), the efficient training duration of agent 2 is $1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}$. Note that T_λ^* is assumed to be an integer, $1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}$ is numeric value and thus not achievable. However, let's first ignore the integer constraint, so that the Planner's objective changes with the graduation date of agent 1 T_1 as

$$\delta^{T_1}\left[\lambda\frac{1+\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}}{1-\delta}+(T_1-1)+\left(1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}-1\right)\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}\right]f(1).$$

At optimality, $T_1=\max\left\{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}+\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta},1\right\}$, which is shorter than the efficient graduation date when there is only a single agent. When $1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}+\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta}\geq 1$, the corresponding value of the Planner's objective is $-\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta}e^{\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e}}f(1)$. When $1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}+\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta}<1$, $T_1=1$, and the corresponding profits are $\delta\left[\lambda\frac{1+\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}}{1-\delta}+\left(1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}-1\right)\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}\right]f(1)$. As a summary, given the integer constraints are satisfied, the maximum value of the Planner's objective under different scenarios is:

[27] The first order condition gives that the optimal T_1 is $T_\lambda^*+\left(5-2\delta-\frac{6}{1+\delta}\right)$, and $0<5-2\delta-\frac{6}{1+\delta}<1$, given that $T_\lambda^*\geq 3$ (which requires $\delta>e^{-\frac{1}{2}}$). The difference in the Planner's objective between contracts with $T_1=T_\lambda^*$ and $T_1=T_\lambda^*+1$ is $-\frac{\delta^{1-\frac{2\lambda}{1-\delta^2}}(1+\delta)}{2e\ln\delta}\left[1-\delta^2-\left(1-\delta(5-2(2-\delta)\delta)\right)\ln\delta\right]>0$.

(i) when agent 2 is trained after agent 1's graduation, the objective is are no larger than $W_0^1 = -\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta} e^{\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e}} f(1)$ or $W_0^1 = \delta\left[\lambda\frac{1+\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}}{1-\delta} + \left(1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}-1\right)\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}\right] f(1)$.

(ii) when agent 2 is trained before agent 1's graduation and T^* is odd, $W_0^2 = \delta^{T_\lambda^*}\left[\lambda\frac{1+\delta}{1-\delta} + (1+\delta)^2\frac{T_\lambda^*-1}{2}\right] f(1)$.

(iii) when agent 2 is trained before agent 1's graduation and T^* is even, $W_0^3 = \delta^{T_\lambda^*}\left[\lambda\frac{1+\delta}{1-\delta} + (1+\delta)^2\left(\frac{T_\lambda^*}{2}-1\right) + 1 + \delta^3\right] f(1)$.

Ignoring the integer constraints and treating W_0^1, W_0^2, and W_0^3 as a continuous function of δ, it can be shown that given $T_\lambda^* \geq 3$,

Suppose $1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}-+\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta} > 1$ so that $W_0^1 = -\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta} e^{\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e}} f(1)$, $\frac{W_0^3}{W_0^1} = \frac{1}{2} e^{-\frac{\delta^{1+\frac{\lambda}{-1+\delta}}}{e}} \delta^{-\frac{\lambda}{1+\delta}}(1+\delta)[1+\delta-(1-(3-2\delta)\delta)\ln\delta]$ is increasing in δ. Therefore, given a fixed λ, $\frac{W_0^3}{W_0^1}$ is smallest when δ is small enough such that the constraint $T_\lambda^* = 3$ binds (thus $\lambda = -\frac{(1-\delta^2)(1+2\ln\delta)}{2\ln\delta}$), and the value is $\frac{1}{2} e^{\frac{1}{2}-\frac{\delta}{2}-e^{\frac{1}{2}(-1+\delta)}\delta^{2+\delta}} \delta^{1-\delta}(1+\delta)(1+\delta-(1-\delta)(1-2\delta)\ln\delta)$, which is greater than 1 when $\delta < \delta^* \approx 0.845$, where δ^* is the solution to the previous equation. The weight λ corresponding to δ^* is about 0.564. When $\lambda > 0.564$, the $1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}-+\frac{\delta^{1-\frac{\lambda}{1-\delta}}}{e\ln\delta} < 1$ so the efficient graduation date of agent 1 is $T_1 = 1$ when agent 2 is trained after agent 1 graduates, and $W_0^1 = \delta\left[\lambda\frac{1+\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}}{1-\delta} + \left(1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}-1\right)\delta^{1-\frac{1}{\ln\delta}-\frac{\lambda}{1-\delta}}\right] f(1)$. It can be shown that $\frac{W_0^3}{W_0^1} > 1$. Therefore, given $T_\lambda^* \geq 3$, $W_0^3 > W_0^1$. Since $W_0^2 - W_0^3 = \frac{\delta^{1-\frac{2\lambda}{1-\delta^2}}\left(-1+\delta\left(2+\delta-2\delta^2\right)\right)}{2e} > 0$ given $T_\lambda^* \geq 3$, we have $W_0^2 > W_0^1$. ■

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